Engineering Maths First Aid Kit

2.26

Completing the square

Introduction

On this leaflet we explain a procedure called **completing the square**. This can be used to solve quadratic equations, and is also important in the calculation of some integrals and when it is necessary to find inverse Laplace transforms.

1. Perfect squares

Some quadratic expressions are **perfect squares**. For example

$$x^2 - 6x + 9 \qquad \text{can be written as} \qquad (x-3)^2$$

The equivalence of this pair of expressions is easily verified by squaring $(x-3)$, as in

$$(x-3)(x-3) = x^2 - 3x - 3x + 9 = x^2 - 6x + 9$$

Similarly, $x^2 + 14x + 49$ can be written as $(x+7)^2$. Both $x^2 - 6x + 9$ and $x^2 + 14x + 49$ are perfect squares because they can be written as the square of another expression.

2. Completing the square

In general, a quadratic expression cannot be written in the form $(***)^2$ and so will not be a perfect square. Often, the best we can do is to write a quadratic expression as a perfect square, plus or minus some constant as you will see in the following example. Doing this is called **completing the square**.

Example
Show that $x^2 + 8x + 7$ can be written as $(x+4)^2 - 9$.

Solution
Squaring the term $(x+4)$ we find

$$\begin{aligned}(x+4)^2 &= (x+4)(x+4)\\ &= x^2 + 8x + 16\end{aligned}$$

So

$$\begin{aligned}(x+4)^2 - 9 &= x^2 + 8x + 16 - 9\\ &= x^2 + 8x + 7\end{aligned}$$

We have shown that $x^2 + 8x + 7$ can be written as a perfect square minus a constant, that is $(x + 4)^2 - 9$. We have completed the square. The following result may help you complete the square, although with practice it is easier to do this by inspection.

$$x^2 + kx + c = (x + \frac{k}{2})^2 - \frac{k^2}{4} + c$$

You can verify this is true by squaring the term in brackets and simplifying the right hand side.

Example
Complete the square for the expression $x^2 + 6x + 2$.

Solution
Comparing $x^2 + 6x + 2$ with the general form in the box above we note that $k = 6$ and $c = 2$. Then

$$\begin{aligned} x^2 + 6x + 2 &= (x + \frac{6}{2})^2 - \frac{6^2}{4} + 2 \\ &= (x + 3)^2 - 7 \end{aligned}$$

and we have completed the square.

Example
Complete the square for the expression $x^2 - 7x + 3$.

Solution
Comparing $x^2 - 7x + 3$ with the general form in the box above we note that $k = -7$ and $c = 3$. Then

$$\begin{aligned} x^2 - 7x + 3 &= (x + \frac{-7}{2})^2 - \frac{(-7)^2}{4} + 3 \\ &= (x - \frac{7}{2})^2 - \frac{49}{4} + 3 \\ &= (x - \frac{7}{2})^2 - \frac{37}{4} \end{aligned}$$

and we have completed the square.

Exercises
1. Complete the square for a) $x^2 - 8x + 5$, b) $x^2 + 12x - 7$.

2. Completing the square can be used in the solution of quadratic equations. Complete the square for $x^2 + 8x + 1$ and use your result to solve the equation $x^2 + 8x + 1 = 0$.

3. By first extracting a factor of 3, complete the square for $3x^2 + 6x + 11$.

Answers
1. a) $(x - 4)^2 - 11$, b) $(x + 6)^2 - 43$.

2. $(x + 4)^2 - 15$. Hence the equation can be written $(x + 4)^2 - 15 = 0$ from which $(x + 4)^2 = 15$, $(x + 4) = \pm\sqrt{15}$ and finally $x = -4 \pm \sqrt{15}$.

3. $3x^2 + 6x + 11 = 3[x^2 + 2x + \frac{11}{3}] = 3[(x + 1)^2 + \frac{8}{3}]$.

